

A NOTE ON THE CHARACTERIZATION OF FINITE BLASCHKE PRODUCTS

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ABSTRACT. We give a slight generalization of the characterization of finite Blaschke products given in [9]. The characterization uses the boundary behaviour of a weighted local hyperbolic distortion of an analytic self-map of the unit disk.

The main purpose of this note is to provide a slight generalization of the characterization of finite Blaschke products given in [9]. The new result includes a wider range of weights than the original one. The notation and the main methods follow [9]. For completeness, we include a short list of definitions and basic concepts used in the statement and in the proof of the result.

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} . For a non-constant analytic function ϕ that maps the unit disk into itself and for $\alpha > 0$, let

$$\tau_{\phi,\alpha}(z) = \frac{(1 - |z|^2)^\alpha |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha}.$$

We will say that $\tau_{\phi,\alpha}(z)$ is the local α -hyperbolic distortion of ϕ at z .

The motivation for this definition comes from the classical hyperbolic case with $\alpha = 1$. Recall that for $z \in \mathbb{D}$, $\lambda(z) = \frac{1}{1 - |z|^2}$ is the density of the hyperbolic metric on \mathbb{D} , and that for an analytic, non-constant map $\phi : \mathbb{D} \rightarrow \mathbb{D}$, the pull-back of the hyperbolic metric is defined by $\phi(\lambda)^*(z) = \frac{|\phi'(z)|}{1 - |\phi(z)|^2}$. Thus,

$$\tau_\phi(z) = \tau_{\phi,1}(z) = \frac{(1 - |z|^2) |\phi'(z)|}{(1 - |\phi(z)|^2)} = \frac{\phi(\lambda)^*(z)}{\lambda(z)}$$

is the usual local hyperbolic distortion of ϕ at z . Similarly, for $\alpha > 0$, we can think of $\tau_{\phi,\alpha}(z)$, the local α -hyperbolic distortion of ϕ at z , as the pull-back by ϕ of the α -hyperbolic metric on \mathbb{D} with density $\lambda_\alpha(z) = \frac{1}{(1 - |z|^2)^\alpha}$. (See, for example, [2] for further details on these and other related basic notions, results and references).

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By the classical Schwarz-Pick lemma, $\tau_\phi(z) \leq 1$ for all $z \in \mathbb{D}$, i.e. every self-map of the unit disk is a hyperbolic contraction. Furthermore, if the equality holds for one $z \in \mathbb{D}$, then ϕ is a disk automorphism, and so the equality must hold for every $z \in \mathbb{D}$. Thus, the maximal possible hyperbolic distortion is attained in the disk only when the map is a disk automorphism. The situation is very different when $\alpha \neq 1$. For example, there are analytic self-maps of \mathbb{D} for which $\tau_{\phi,\alpha}(z)$ is not even bounded when $0 < \alpha < 1$. For more details and further references on this, see [9].

In 1986, M. Heins used in [4] the boundary behaviour of the hyperbolic distortion τ_ϕ to characterize the finite Blaschke products among the class of analytic self-maps of the unit disk.

Theorem A. ([4]): Let ϕ be an analytic self-map of \mathbb{D} . Then ϕ is a finite Blaschke product if and only if $\lim_{|z| \rightarrow 1} \tau_\phi(z) = 1$.

A more recent result of D. Kraus, O. Roth and S. Rucheweyh from 2007 generalized Heins' result to a characterization of analytic boundary behaviour of self-maps of \mathbb{D} on subarcs of the unit circle.

Theorem B. ([6]): Let ϕ be an analytic self-map of \mathbb{D} and let Γ be an open subarc of $\partial\mathbb{D}$. Then the following are equivalent:

- (a) For every $\zeta \in \Gamma$, $\liminf_{z \rightarrow \zeta} \tau_\phi(z) > 0$.
- (b) For every $\zeta \in \Gamma$, $\lim_{z \rightarrow \zeta} \tau_\phi(z) = 1$.
- (c) ϕ has an analytic extension across Γ and $\phi(\Gamma) \subset \partial\mathbb{D}$.

They also asked questions about possible generalizations of these results in terms of angular, i.e. non-tangential, limits. Recall that for $\zeta \in \partial\mathbb{D}$ and $\gamma > 1$, the angular (or non-tangential) region $\Gamma_\gamma(\zeta)$ in \mathbb{D} is defined by

$$\Gamma_\gamma(\zeta) = \{z \in \mathbb{D} : |\zeta - z| \leq \gamma(1 - |z|^2)\}.$$

If $z \rightarrow \zeta$ through the angular region $\Gamma_\gamma(\zeta)$, we say that the corresponding limit is an *angular limit*. This type of limit will be denoted by $\angle \lim_{z \rightarrow \zeta}$. Recall that since an analytic self-map ϕ of the unit disk is in $H^\infty(\mathbb{D})$, it has radial (and angular) limits almost everywhere on the unit circle. We will denote the radial extension function with the same symbol ϕ .

Moreover, an analytic map $\phi : \mathbb{D} \rightarrow \mathbb{D}$ has an *angular derivative* at $\zeta \in \partial\mathbb{D}$ if there exists $\xi \in \partial\mathbb{D}$ such that the angular limit

$$\angle \lim_{z \rightarrow \zeta} \frac{\phi(z) - \xi}{z - \zeta}$$

exists. In this case the value of the limit is called an angular derivative of ϕ at ζ , and will be denoted by $\phi'(\zeta)$. By the Julia-Carathéodory theorem (see [8]), the existence of the angular derivative at ζ is equivalent to

$$0 < \liminf_{z \rightarrow \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = |\phi'(\zeta)| < \infty,$$

with the limit infimum attained in an angular approach to ζ .

In 2013, D. Kraus specified in [5] the class of functions characterized by the existence of (nonzero) angular boundary limits of their hyperbolic distortion a.e. on the unit circle.

Theorem C.([5]): If ϕ is an analytic self-map of \mathbb{D} , then $\angle \lim_{z \rightarrow \zeta} \tau_\phi(z) = 1$ for almost every $\zeta \in \partial\mathbb{D}$ if and only if ϕ is an inner function with finite angular derivatives at almost every point in $\partial\mathbb{D}$.

Since there exist infinite Blaschke products with finite angular derivatives a.e. on $\partial\mathbb{D}$, the class of functions determined by the last theorem is much larger than the class of finite Blaschke products. Hence, the angular limits boundary condition on the classical hyperbolic distortion τ_ϕ will not determine the class of finite Blaschke products. We were able to show in [9] that we can accomplish this if, instead, we use angular limit boundary conditions on $\tau_{\phi,\alpha}$, with $\alpha > 1$. The following theorem shows that actually, this result can be generalized to any $\alpha \neq 1$. Parts of the proof of the theorem are the same as in [9]. We provide some of them here for completeness.

Theorem 1. Let ϕ be a non-constant self-map of \mathbb{D} . Then ϕ is a finite Blaschke product if and only if there exist $\alpha \neq 1$ and $c > 0$ such that $\tau_{\phi,\alpha}$ is bounded and such that for almost every $\zeta \in \partial\mathbb{D}$

$$\angle \liminf_{z \rightarrow \zeta} \tau_{\phi,\alpha}(z) \geq c.$$

Proof. If ϕ is a finite Blaschke product, then $\tau_{\phi,\alpha}$ is bounded for any $\alpha > 0$ as shown, for example, in [9]. Since ϕ has an analytic extension across the unit circle, and $|\phi(\zeta)| = 1$ for all $\zeta \in \partial\mathbb{D}$, by the Julia-Carathéodory theorem ϕ has an angular derivative equal to the regular derivative of ϕ at ζ . Using again the Julia-Carathéodory theorem characterization of the angular derivative, one gets that

$$\angle \lim_{z \rightarrow \zeta} \tau_{\phi,\alpha}(z) = |\phi'(\zeta)|^{1-\alpha}.$$

If $\alpha > 1$, since ϕ' is continuous on $\overline{\mathbb{D}}$, we get that

$$\angle \lim_{z \rightarrow \zeta} \tau_{\phi,\alpha}(z) = |\phi'(\zeta)|^{1-\alpha} \geq \|\phi'\|_\infty^{1-\alpha} > 0.$$

If $0 < \alpha < 1$, since by the Schwartz Lemma $\frac{1-|\phi(z)|^2}{1-|z|^2} \geq \frac{1-|\phi(0)|}{1+|\phi(0)|}$, we get that

$$\angle \lim_{z \rightarrow \zeta} \tau_{\phi,\alpha}(z) = |\phi'(\zeta)|^{1-\alpha} = \angle \lim_{z \rightarrow \zeta} \left(\frac{1-|\phi(z)|^2}{1-|z|^2} \right)^{1-\alpha} \geq \left(\frac{1-|\phi(0)|}{1+|\phi(0)|} \right)^{1-\alpha} > 0.$$

For the other direction of the proof, we will show first, by using a similar idea as in the proof of Theorem 2.3 from [5], that if $\exists \alpha > 0, c > 0$ such that $\angle \liminf_{z \rightarrow \zeta} \tau_{\phi,\alpha}(z) \geq c$ for almost every $\zeta \in \mathbb{D}$, then ϕ has to be an inner function. Note that $1-|z|^2 \rightarrow 0$ as $z \rightarrow \zeta \in \partial\mathbb{D}$, and so for almost all $\zeta \in \partial\mathbb{D}$

$$\angle \lim_{z \rightarrow \zeta} \frac{|\phi'(z)|}{(1-|\phi(z)|)^\alpha} = \infty.$$

Thus, if $\angle \lim_{z \rightarrow \zeta} |\phi(z)| = r < 1$, then $\angle \lim_{z \rightarrow \zeta} |\phi'(z)| = \infty$. But by Privalov's Theorem [7, p. 140], this can happen only for ζ in a set of measure zero, i.e. $\angle \lim_{z \rightarrow \zeta} |\phi(z)| = 1$ for almost every $\zeta \in \partial\mathbb{D}$, and so ϕ is inner.

In the case when $0 < \alpha < 1$, since ϕ is inner and $\tau_{\phi, \alpha}$ is bounded, ϕ must be a finite Blaschke product. This follows, for example, from the fact that then ϕ has to be in the disk algebra (see [3, p. 74]).

When $\alpha > 1$, and for almost every $\zeta \in \partial\mathbb{D}$

$$\angle \liminf_{z \rightarrow \zeta} \tau_{\phi, \alpha}(z) = \angle \liminf_{z \rightarrow \zeta} \left(\frac{1 - |z|^2}{1 - |\phi(z)|^2} \right)^{\alpha-1} \tau_{\phi}(z) \geq c > 0,$$

then, since by the Schwarz-Pick Lemma $\tau_{\phi}(z) \leq 1$ for every $z \in \mathbb{D}$, and since $\alpha - 1 > 0$, we have that

$$\angle \liminf_{z \rightarrow \zeta} \frac{1 - |z|^2}{1 - |\phi(z)|^2} \geq c^{\frac{1}{\alpha-1}}.$$

Thus, for almost every ζ in $\partial\mathbb{D}$

$$\angle \limsup_{z \rightarrow \zeta} \frac{1 - |\phi(z)|^2}{1 - |z|^2} \leq \left(\frac{1}{c} \right)^{\frac{1}{\alpha-1}},$$

and so, for almost every ζ in $\partial\mathbb{D}$ it must be that

$$|\phi'(\zeta)| = \liminf_{z \rightarrow \zeta} \frac{1 - |\phi(z)|}{1 - |z|} \leq 2 \left(\frac{1}{c} \right)^{\frac{1}{\alpha-1}} < \infty.$$

Hence, we have that on one hand ϕ is inner, as shown above, and on the other hand for almost every ζ in $\partial\mathbb{D}$ the modulus of the angular derivative at ζ is bounded above by a constant depending only on α and c .

By a result of P. Ahern and D. Clark (see [1]), for an inner function ϕ we have that $|\phi'|$ is in $L^p(\partial\mathbb{D})$, $0 < p \leq \infty$ if and only if ϕ' is in $H^p(\mathbb{D})$. Furthermore, if $\phi' \in H^1(\mathbb{D})$, then ϕ must be a finite Blaschke product. Since in our case $|\phi'|$ is almost everywhere bounded by $2 \left(\frac{1}{c} \right)^{\frac{1}{\alpha-1}}$ on $\partial\mathbb{D}$, by [1] we have that ϕ' belongs to $H^\infty(\mathbb{D})$, and so ϕ must be a finite Blaschke product. \square

Remark: Note that as a consequence of the theorem, one also gets an interesting result reflecting on the rigidity of the local hyperbolic distortion. Namely, if there exist $\alpha \neq 1$ and $c > 0$ such that $\tau_{\phi, \alpha}$ is bounded and such that for almost every $\zeta \in \partial\mathbb{D}$ we have that $\angle \liminf_{z \rightarrow \zeta} \tau_{\phi, \alpha}(z) \geq c$, then $\angle \lim_{z \rightarrow \zeta} \tau_{\phi, \alpha}(z)$ exists for every ζ in $\partial\mathbb{D}$ and equals $|\phi'(\zeta)|^{1-\alpha}$. Also, the proof of the theorem shows that if in the previous statement we replace $\partial\mathbb{D}$ by a subset E of positive Lebesgue (linear) measure, then $\angle \lim_{z \rightarrow \zeta} |\phi(z)| = 1$ for almost every $\zeta \in E$ and the conclusion on the rigidity of $\tau_{\phi, \alpha}$ holds almost everywhere on E .

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